

# ON FUJITA INVARIANTS OF SUBVARIETIES OF A UNIRULED VARIETY

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**ABSTRACT.** We show that if  $X$  is a smooth uniruled projective variety and  $L$  a big and semiample  $\mathbb{Q}$ -divisor on  $X$ , then there exists a proper closed subset  $W \subset X$  such that every subvariety  $Y$  satisfying  $a(Y, L) > a(X, L)$  is contained in  $W$ .

## 1. INTRODUCTION

If  $X$  is a smooth projective variety and  $L$  is a big  $\mathbb{Q}$ -divisor on  $X$ , then the *Fujita invariant*, or *a-constant* is defined as follows

$$a(X, L) = \inf\{t > 0 \mid K_X + tL \text{ is big}\}.$$

Note that  $a(X, L) \in \mathbb{R}_{\geq 0}$  is well defined since  $K_X + tL$  is big for all  $t > 0$  sufficiently large, and that  $a(X, L) > 0$  if and only if  $K_X$  is not pseudo-effective. It is easy to see that the  $a$ -constant is a birational invariant in the sense that if  $\nu : X' \rightarrow X$  is a birational morphism of smooth varieties and  $L' = \nu^*L$ , then  $a(X, L) = a(X', L')$ . Therefore we may also define the  $a$ -constant for a big  $\mathbb{Q}$ -Cartier  $\mathbb{Q}$ -divisor  $L$  on an arbitrary normal projective variety  $X$  by letting

$$a(X, L) := a(X', L')$$

where  $\nu : X' \rightarrow X$  is a resolution of singularities and  $L' = \nu^*L$ . Note that if  $X$  is smooth, then the  $a$ -constant is the usual pseudo-effective threshold, however if  $X$  is singular, these numbers may be different.

In [8], motivated by a conjecture of Batyrev and Manin that relates arithmetic properties of varieties with ample anticanonical class to geometric invariants,  $a$ -constants were intensively studied by Lehmann, Tanimoto and Tschinkel. They show that ([8, Theorem 1.1]), if  $X$  is a smooth uniruled projective variety and  $L$  an ample  $\mathbb{Q}$ -divisor on  $X$ , then there exists a countable union of proper closed subsets  $W \subset X$  such that every subvariety  $Y$  satisfying  $a(Y, L) > a(X, L)$  is contained in  $W$ . For the purpose of applications, it is expected that one may choose  $W$  to be a proper closed subset of  $X$ . The purpose of this note is to prove that this is indeed the case:

**Theorem 1.1.** *Let  $X$  be a smooth uniruled projective variety and  $L$  a big and semiample  $\mathbb{Q}$ -divisor on  $X$ . Then there exists a proper closed subset  $W \subset X$  such that every subvariety  $Y$  satisfying  $a(Y, L) > a(X, L)$  is contained in  $W$ .*

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Note that this result is proven in [8, Theorem 1.2] assuming that a weak version of the BAB conjecture holds in dimension  $n - 1 = \dim X - 1$ . We expect that Theorem 1.1 holds also if we just assume that  $L$  is nef and big (rather than big and semiample).

Our idea is to replace the WBAB conjecture assumed in [8, Theorem 1.2] by constructing non-klt centers (see Proposition 2.8) and applying finiteness of  $a$ -constants (see Corollary 2.15). This is an application of a recent result of Di Cerbo [3] based on a boundedness result proved by Birkar [2].

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## 2. PRELIMINARIES

In this paper we work over the field of complex numbers  $\mathbb{C}$ .

**2.1. Facts on  $a$ -constants.** In this subsection, for the convenience of the reader, we collect several facts about  $a$ -constants that were proven in [8].

**Proposition 2.1** ([8, Proposition 4.1]). *Let  $X$  be a smooth projective variety and  $L$  a big and nef  $\mathbb{Q}$ -divisor. Let  $\mathcal{U} \rightarrow W$  be a family of subvarieties of  $X$  such that  $\mathcal{U} \rightarrow X$  is dominant. Then a general member  $Y$  of the family  $\mathcal{U}$  satisfies  $a(Y, L) \leq a(X, L)$ .*

**Theorem 2.2** ([8, Theorem 4.2]). *Let  $X$  be a smooth projective variety and  $L$  a big and nef  $\mathbb{Q}$ -divisor. Let  $\pi : \mathcal{U} \rightarrow W$  be a family of subvarieties of  $X$ . There exists a proper closed subset  $V \subset X$  such that if a member  $Y$  of the family  $\mathcal{U}$  satisfies  $a(Y, L) > a(X, L)$  then  $Y \subset V$ .*

**Proposition 2.3** ([8, Proposition 4.6]). *Let  $X$  be a smooth uniruled projective variety and  $L$  a big and nef  $\mathbb{Q}$ -divisor. Then either*

- (1)  $X$  is covered by proper subvarieties  $Y$  satisfying  $a(Y, L) = a(X, L)$ ,  
or
- (2)  $X$  is birational to a  $\mathbb{Q}$ -factorial terminal Fano variety  $X'$  of Picard number 1.

**Lemma 2.4** ([8, Lemma 4.7]). *Let  $X$  be a smooth projective variety and  $L$  a big and nef  $\mathbb{Q}$ -divisor on  $X$ . Fix a constant  $C$ . Then the subset of  $\text{Chow}(X)$  parametrizing subvarieties of  $X$  that are not contained in  $\mathbf{B}_+(L)$  and are of  $L$ -degree at most  $C$  is bounded.*

**2.2. Non-klt centers.** We follow the standard notation and conventions of the minimal model program, see eg. [5].

**Definition 2.5.** Let  $(X, \Delta)$  be a pair so that  $X$  is a normal variety,  $\Delta$  is an effective  $\mathbb{Q}$ -divisor, and  $K_X + \Delta$  is  $\mathbb{Q}$ -Cartier. We say that a subvariety  $V \subset X$  is a *non-klt center* of  $(X, \Delta)$  if it is the image of a divisor of discrepancy at most  $-1$ . We will denote by  $\text{Nklt}(X, \Delta)$  the union of all non-klt centers of  $(X, \Delta)$ . A *non-klt place* is a valuation corresponding to a divisor of discrepancy at most  $-1$ . A non-klt center is *pure* if  $K_X + \Delta$  is log canonical at the generic point of  $V$ . If moreover there is a unique non-klt

place lying over the generic point of  $V$ , we will say that  $V$  is an *exceptional non-klt center*.

The following is a weak form of Kawamata's subadjunction theorem.

**Theorem 2.6** (Subadjunction, see [4, Proposition 5.1]). *Let  $V \subset X$  be a non-klt center of a pair  $(X, \Delta)$  which is lc at a general point of  $V$ . Let  $\nu : V^\nu \rightarrow V$  be the normalization. Then there is an effective  $\mathbb{Q}$ -divisor  $\Delta_{V^\nu}$  on  $V^\nu$  such that*

$$\nu^*(K_X + \Delta)|_{V^\nu} \sim_{\mathbb{Q}} K_{V^\nu} + \Delta_{V^\nu}.$$

We have the following connectedness lemma of Kollár and Shokurov for the non-klt locus (cf. Shokurov [9], Kollár [6, 17.4]).

**Theorem 2.7** (Connectedness Lemma). *Let  $f : X \rightarrow Z$  be a proper morphism of normal varieties with connected fibers and  $D$  a  $\mathbb{Q}$ -divisor such that  $-(K_X + D)$  is  $\mathbb{Q}$ -Cartier,  $f$ -nef, and  $f$ -big. Write  $D = D^+ - D^-$  where  $D^+$  and  $D^-$  are effective with no common components. If  $D^-$  is  $f$ -exceptional (i.e. all of its components have image of codimension at least 2), then  $\text{Nklt}(X, D) \cap f^{-1}(z)$  is connected for any  $z \in Z$ .*

We can use the following proposition to construct non-klt centers.

**Proposition 2.8** (cf. [7, Lemma 3.2]). *Let  $X$  be a  $\mathbb{Q}$ -factorial terminal Fano variety of dimension  $n$ . Assume  $(-K_X)^n > (wn)^n$  for some positive rational number  $w$ . Then for every point  $p \in X$  there is an effective  $\mathbb{Q}$ -divisor  $\Delta_p \sim_{\mathbb{Q}} -\frac{1}{w}K_X$  such that the unique minimal non-klt center  $V_p \subset \text{Nklt}(X, \Delta_p)$  containing  $p$  is exceptional.*

*Proof.* Fix a point  $p$ . Fix a positive rational number  $w'$  such that  $(-K_X)^n > (w'n)^n > (wn)^n$ . By [5, 6.7.1 Theorem], there is an effective  $\mathbb{Q}$ -divisor  $\Delta'_p \sim_{\mathbb{Q}} -\frac{1}{w'}K_X$  such that  $(X, \Delta'_p)$  is not lc at  $p$ . Take  $0 < t \leq 1$  the unique rational number such that  $(X, t\Delta'_p)$  is log canonical but not klt at  $p$ . By [1, Proposition 3.2, Lemma 3.4], we can find an effective  $\mathbb{Q}$ -divisor  $M_p \sim_{\mathbb{Q}} -\frac{1}{w}K_X$  and some rational number  $a > 0$  such that for any rational number  $0 < \epsilon \ll 1$ , the pair  $(X, (1 - \epsilon)t\Delta'_p + \epsilon a M_p)$  has a unique minimal non-klt center  $V_p$  passing through  $p$  which is exceptional. Note that

$$(1 - \epsilon)t\Delta'_p + \epsilon a M_p \sim_{\mathbb{Q}} -\frac{(1 - \epsilon)t + \epsilon a}{w'}K_X$$

and  $\frac{(1 - \epsilon)t + \epsilon a}{w'} < \frac{1}{w}$  for  $0 < \epsilon \ll 1$ . Since  $-K_X$  is ample, by adding a  $\mathbb{Q}$ -divisor  $\mathbb{Q}$ -linearly equivalent to a multiple of  $-K_X$  to  $\Delta'_p$ , we conclude that there exists an effective  $\mathbb{Q}$ -divisor  $\Delta_p \sim_{\mathbb{Q}} -\frac{1}{w}K_X$  and  $(X, \Delta_p)$  has a unique minimal non-klt center  $V_p$  passing through  $p$  which is exceptional.  $\square$

**Lemma 2.9.** *Keep the notation in Proposition 2.8. If  $w > 2$ , then  $\dim V_p > 0$  for a general point  $p$ .*

*Proof.* Assume to the contrary that there exist  $p_1 \in X$  such that  $V_{p_1} = \{p_1\}$  and  $p_2 \in X \setminus \text{Supp}(\Delta_{p_1})$  such that  $V_{p_2} = \{p_2\}$ . Then  $p_1$  and  $p_2$  are contained in  $\text{Nklt}(X, \Delta_{p_1} + \Delta_{p_2})$  and  $p_2$  is isolated by construction. On the other hand,

$$-(K_X + \Delta_{p_1} + \Delta_{p_2}) \sim_{\mathbb{Q}} \left(1 - \frac{2}{w}\right)(-K_X)$$

is ample. By the connectedness lemma,  $\text{Nklt}(X, \Delta_{p_1} + \Delta_{p_2})$  is connected, which is a contradiction.  $\square$

**2.3. Finiteness of  $a$ -constants.** We recall the main result of [3] in this subsection.

**Definition 2.10.** Let  $X$  be a normal projective variety and  $H$  a big  $\mathbb{Q}$ -divisor. We define the *pseudo-effective threshold* to be

$$\tau(X, H) := \inf\{t \geq 0 \mid K_X + tH \text{ is big}\}.$$

Note that if  $X$  is smooth,  $a$ -constant and pseudo-effective threshold just coincide.

**Definition 2.11** (cf. [3, Definition 3.1]). Fix a positive integer  $n$  and two positive real numbers  $\epsilon$  and  $\delta$ . We define  $\mathcal{D}_n(\epsilon, \delta)$  to be the set of lc pairs  $(X, \Delta)$  such that:

- (1)  $X$  is a normal projective variety of dimension  $n$ ,
- (2)  $\Delta$  is a big  $\mathbb{Q}$ -divisor with coefficients  $\geq \delta$ , and
- (3)  $(X, t\Delta)$  is  $\epsilon$ -lc and  $K_X + t\Delta$  is pseudo-effective for some  $0 \leq t \leq 1$ .

**Definition 2.12** (cf. [3, Definition 3.2]). Fix a positive integer  $n$  and two positive real numbers  $\epsilon$  and  $\delta$ . We define the set

$$\mathcal{T}_n(\epsilon, \delta) := \{\tau(X, \Delta) \mid (X, \Delta) \in \mathcal{D}_n(\epsilon, \delta)\}.$$

**Theorem 2.13** ([3, Corollary 3.6]). *Fix a positive integer  $n$  and three positive real numbers  $\epsilon, \delta$  and  $\eta$ . Then the set  $\mathcal{T}_n(\epsilon, \delta) \cap [\eta, 1]$  is a finite set.*

To apply this theorem in our situation, we have the following corollary.

**Definition 2.14.** Fix a positive integer  $n$ . We define  $\mathcal{P}_n$  to be the set of pairs  $(Y, L)$  such that:

- (1)  $Y$  is a normal projective variety of dimension  $n$ ,
- (2)  $L$  is a base point free big Cartier divisor.

**Corollary 2.15.** *Fix a positive integer  $n$  and a positive real number  $\eta$ . Then the set*

$$\{a(Y, L) \mid (Y, L) \in \mathcal{P}_n\} \cap [\eta, \infty)$$

*is a finite set.*

*Proof.* We may assume that  $\eta \leq \frac{1}{4(n+1)}$ .

Firstly, we show that the set

$$\{a(Y, L) \mid (Y, L) \in \mathcal{P}_n\} \cap \left[\eta, \frac{1}{2}\right]$$

is a finite set. Take  $(Y, L) \in \mathcal{P}_n$  and assume that  $a(Y, L) \in [\eta, \frac{1}{2}]$ . Note that  $a(Y, \frac{1}{2}L) = 2a(Y, L) \in [2\eta, 1]$ . By taking a resolution, we may assume that  $Y$  is smooth. In this case  $a(Y, \frac{1}{2}L) = \tau(Y, \frac{1}{2}L)$ . Replacing  $L$  by a general element in  $|L|$ , we may assume that  $L$  is irreducible and smooth. Moreover,  $(Y, \frac{1}{2}L)$  is  $\frac{1}{2}$ -lc and  $K_Y + \frac{1}{2}L$  is pseudo-effective, that is,  $(Y, \frac{1}{2}L) \in \mathcal{D}_n(\frac{1}{2}, \frac{1}{2})$ . This implies that the set

$$\left\{a\left(Y, \frac{1}{2}L\right) \mid (Y, L) \in \mathcal{P}_n\right\} \cap [2\eta, 1]$$

is finite by Theorem 2.13, and so is  $\{a(Y, L) \mid (Y, L) \in \mathcal{P}_n\} \cap [\eta, \frac{1}{2}]$ .

Then we show that the set

$$\{a(Y, L) \mid (Y, L) \in \mathcal{P}_n\} \cap \left[\frac{1}{2}, \infty\right)$$

is a finite set. Take  $(Y, L) \in \mathcal{P}_n$  and assume that  $a(Y, L) \geq \frac{1}{2}$ . By taking a resolution, we may assume that  $Y$  is smooth. By [8, Proposition 2.10],  $a(Y, L) \leq n+1$ . Now we consider  $(Y, 2(n+1)L) \in \mathcal{P}_n$ . Note that  $a(Y, 2(n+1)L) = \frac{1}{2(n+1)}a(Y, L)$ , hence  $a(Y, 2(n+1)L) \in [\frac{1}{4(n+1)}, \frac{1}{2}]$ . By the first step,  $a(Y, 2(n+1)L)$  belongs to a finite set. Hence  $a(Y, L)$  belongs to a finite set.  $\square$

### 3. PROOF OF THEOREM 1.1

We prove the following proposition suggested by B. Lehmann.

**Proposition 3.1.** *Fix a positive real number  $t$ . Let  $X$  be a smooth projective variety and  $L$  a big and semiample  $\mathbb{Q}$ -divisor. Then there is a bounded family  $\mathcal{U}$  of subvarieties of  $X$  such that any subvariety  $Y$  not contained in  $\mathbf{B}_+(L)$ , with  $a(Y, L) > t$  is dominated by some members  $Z$  of  $\mathcal{U}$ , such that  $a(Z, L) = a(Y, L)$ .*

*Proof.* Note that for a subvariety  $Y$  not contained in  $\mathbf{B}_+(L)$ ,  $L|_Y$  is nef and big, and so  $a(Y, L)$  is well defined. Therefore we will only consider subvarieties not contained in  $\mathbf{B}_+(L)$ .

Replacing  $L$  by some multiple, we may assume that  $L$  is a base point free Cartier divisor.

We construct  $\mathcal{U}$  inductively by increasing induction on the dimension of  $Y$ .

For a subvariety  $Y$  with  $a(Y, L) > t$  and  $\dim Y = 1$ , it is easy to see that  $Y$  is a rational curve with

$$\deg_Y(L) = Y \cdot L = \frac{2}{a(Y, L)} < \frac{2}{t}.$$

By Lemma 2.4, such  $Y$  form a bounded family  $\mathcal{U}_1$ .

Suppose that we have constructed a bounded family  $\mathcal{U}_i$  of subvarieties such that every subvariety  $Y$  with  $a(Y, L) > t$  and  $\dim Y \leq i$  is dominated by some members  $Z$  of  $\mathcal{U}$  such that  $a(Z, L) = a(Y, L)$ . We construct  $\mathcal{U}_{i+1}$  as follows. Suppose that  $Y$  is an  $(i+1)$ -dimensional subvariety satisfying  $a(Y, L) > t$ . By taking a resolution, we may assume that  $Y$  is smooth. Proposition 2.3 shows that either

- (1)  $Y$  is covered by proper subvarieties  $Z$  with  $a(Z, L) = a(Y, L)$ , or
- (2)  $Y$  is birational to a  $\mathbb{Q}$ -factorial terminal Fano variety  $Y'$  of Picard number 1.

In Case (1), by induction,  $Z$  is dominated by some members  $Z'$  of  $\mathcal{U}_i$  such that  $a(Z', L) = a(Z, L)$ , and so is  $Y$ .

In Case (2), by taking a resolution, we may assume  $\phi : Y \dashrightarrow Y'$  is a morphism. By the proof of [8, Proposition 4.6],  $K_{Y'} + a(Y, L)\phi_*(L|_Y) \equiv 0$ .

We define constant  $c_0 < 1$  and  $w > 2$  as follows: since  $L$  is base point free, we know that the set

$$\{a(Z, L) \mid Z \text{ is a subvariety of } X\} \cap (t, \infty]$$

is finite by Corollary 2.15. Hence we may take a rational number  $c_0 < 1$  such that the set

$$\{a(Z, L) \mid Z \text{ is a subvariety of } X\} \cap [c_0 a(Z', L), a(Z', L))$$

is empty for any subvariety  $Z'$  with  $a(Z', L) > t$ . Take  $w = \frac{1}{1-c_0}$ . We may assume  $w > 2$  by decreasing  $c_0$ .

If  $(-K_{Y'})^{i+1} \leq (w(i+1))^{i+1}$ , then

$$(L|_Y)^{i+1} \leq (\phi_*(L|_Y))^{i+1} \leq \frac{(w(i+1))^{i+1}}{a(Y, L)^{i+1}} < \frac{(w(i+1))^{i+1}}{a(X, L)^{i+1}}.$$

Then by Lemma 2.4, such  $Y$  form a bounded family  $\mathcal{U}'_{i+1}$ .

Now we assume that  $(-K_{Y'})^{i+1} > (w(i+1))^{i+1}$ . By Proposition 2.8, for a general point  $p \in Y'$ , there exists an effective  $\mathbb{Q}$ -divisor  $\Delta'_p \sim_{\mathbb{Q}} -\frac{1}{w}K_{Y'}$  such that  $V'_p \subset \text{Nklt}(Y', \Delta'_p)$  is the minimal exceptional non-klt center containing  $p$ . Note that by Lemma 2.9 and  $w > 2$ ,  $\dim V'_p > 0$ . Let  $\nu : \tilde{V}_p^\nu \rightarrow V'_p$  be the normalization. For any  $\mathbb{Q}$ -Cartier divisor  $G$  on  $V'_p$ , we denote  $G|_{\tilde{V}_p^\nu} = \nu^*G$ . By Theorem 2.6, there is an effective  $\mathbb{Q}$ -divisor  $\Delta_{\tilde{V}_p^\nu}$  such that

$$(K_{Y'} + \Delta'_p)|_{\tilde{V}_p^\nu} \sim_{\mathbb{Q}} K_{\tilde{V}_p^\nu} + \Delta_{\tilde{V}_p^\nu}.$$

Note that since  $K_{Y'} + a(Y, L)\phi_*L \equiv 0$ , we have

$$K_{\tilde{V}_p^\nu} + \Delta_{\tilde{V}_p^\nu} + \left(1 - \frac{1}{w}\right)a(Y, L)\phi_*L|_{\tilde{V}_p^\nu} \sim_{\mathbb{Q}} 0.$$

Let  $V_p$  be the strict transform of  $V'_p$  on  $Y$ . Let  $\tilde{V}_p$  be a common resolution of  $\tilde{V}_p^\nu$  and  $V_p$ ,  $f : \tilde{V}_p \rightarrow V_p$ ,  $g : \tilde{V}_p \rightarrow \tilde{V}_p^\nu$ . Then

$$\begin{aligned} & K_{\tilde{V}_p} + \left(1 - \frac{1}{w}\right)a(Y, L)f^*(L|_{V_p}) \\ &= g^*\left(K_{\tilde{V}_p^\nu} + \Delta_{\tilde{V}_p^\nu} + \left(1 - \frac{1}{w}\right)a(Y, L)\phi_*L|_{\tilde{V}_p^\nu}\right) - g_*^{-1}\Delta_{\tilde{V}_p^\nu} + E \\ &\sim_{\mathbb{Q}} -g_*^{-1}\Delta_{\tilde{V}_p^\nu} + E, \end{aligned}$$

where  $E$  is a  $g$ -exceptional  $\mathbb{Q}$ -divisor. Note that the  $\mathbb{Q}$ -divisor  $-g_*^{-1}\Delta_{\tilde{V}_p^\nu} + E$  is not big. Hence  $K_{\tilde{V}_p} + \left(1 - \frac{1}{w}\right)a(Y, L)f^*(L|_{V_p})$  is not big and therefore

$$a(V_p, L) \geq \left(1 - \frac{1}{w}\right)a(Y, L) = c_0 a(Y, L).$$

By the definition of  $c_0$ , this implies that  $a(V_p, L) \geq a(Y, L)$ . Since  $p$  is a general point,  $Y$  is dominated by such  $V_p$ . By induction,  $V_p$  is dominated by some members  $Z$  of  $\mathcal{U}_i$  such that  $a(Z, L) = a(V_p, L) \geq a(Y, L)$ . Hence  $Y$  is dominated by some members  $Z$  of  $\mathcal{U}_i$  such that  $a(Z, L) \geq a(Y, L)$ . By Proposition 2.1, by taking general members,  $Y$  is dominated by some members  $Z$  of  $\mathcal{U}_i$  such that  $a(Z, L) = a(Y, L)$ .

Hence we may take  $\mathcal{U}_{i+1} = \mathcal{U}_i \cup \mathcal{U}'_{i+1}$ , and the proof is completed.  $\square$

*Proof of Theorem 1.1.* Take  $t = a(X, L)$  in Proposition 3.1, there is a bounded family  $\mathcal{U}$  of subvarieties of  $X$  such that any subvariety  $Y$  not contained in  $\mathbf{B}_+(L)$ , with  $a(Y, L) > a(X, L)$  is dominated by some members  $Z$  of  $\mathcal{U}$ , such that  $a(Z, L) = a(Y, L) > a(X, L)$ . By Theorem 2.2, there exists a proper

closed subset  $W \subset X$  such that any member  $Z$  of the family  $\mathcal{U}$  satisfying  $a(Z, L) > a(X, L)$  is contained in  $W$ . Hence any subvariety  $Y$  with  $a(Y, L) > a(X, L)$  is contained in  $W$ .  $\square$

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